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On a Conjecture of S. Chowla and of S. Chowla and H. Walum, I

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IN HONOUR OF PROFESSOR S. CHOWLA

As an extension of the Dirichlet divisor problem, S. Chowla and H. Walum conjectured that, as $x \to \infty$, $\sum_{n \leqslant \sqrt{x}} n^a B_r(\{x/n\}) = O(x^{a/2+1/4+\varepsilon})$ holds for each $\varepsilon > 0$. Here integers $a \geqslant 0$ and $r \geqslant 1$ are given. $B_r(x)$ denotes the rth Bernoulli polynomial and $\{x\}$ denotes the fractional part of x. The special case $a = 0$, $r = 2$ of this conjecture was also mentioned by S. Chowla. In this paper we prove this conjecture for all $e \geqslant \frac{1}{2}$ and $r \geqslant 2$ with $\varepsilon = 0$ (with x^ε replaced by log x in case $a = \frac{1}{2}$). © 1985 Academic Press, Inc.

Throughout the paper, x denotes a real variable $\geqslant 1$. $[x]$ denotes the greatest integer $\leqslant x$ and $\{x\} = x - [x]$ the fractional part of x. Let $d(n)$ denote the number of divisors of a positive integer n and let $\Delta(x) = \sum_{n<x} d(n) - x(\log x + 2\gamma - 1)$, where γ is Euler's constant. The classical Dirichlet's divisor problem states that

$$\Delta(x) = O(x^{1/4+\varepsilon}) \tag{1}$$

for every $\varepsilon > 0$. This remains unsettled to date. The best result known till now is due to G. A. Kolesnik [3], who proved that $\Delta(x) = O(x^{346/1067} \cdot \log^{211/100} x)$. As an extension of the Dirichlet divisor problem, S. Chowla and H. Walum [1] (see also S. L. Segal [6]) conjectured the following:

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Let $a \geqslant 0$, $r \geqslant 1$ be integers and $B_r(x)$ denote the rth Bernoulli polynomial. Then for every $\varepsilon > 0$ and as $x \to \infty$

$$\sum_{n \leqslant \sqrt{x}} n^a B_r\left(\left\{\frac{x}{n}\right\}\right) = O(x^{a/2 + 1/4 + \varepsilon}). \tag{2}$$

It is well known due to E. Landau [4] that

$$\Delta(x) + 2 \sum_{n \leqslant \sqrt{x}} B_1\left(\left\{\frac{x}{n}\right\}\right) = O(1).$$

As such, the conjecture (2) in case $a = 0$, $r = 1$ is equivalent to the truth of (1).

The special case $a = 0$, $r = 2$ of (2) was also mentioned by S. Chowla in his book [2] while S. Chowla and H. Walum [1] disposed of the case $a = 1$, $r = 2$ of (2) even with $\varepsilon = 0$. The object of the present paper is to prove the conjecture (2) for all real $a \geqslant \frac{1}{2}$ and integral $r \geqslant 2$ in a stronger form. In fact, writing $G_{a,r}(x, Q) = \sum_{n \leqslant Q} n^a B_r(\{x/n\})$, we prove the following results:

THEOREM 1. *Let $r \geqslant 2$, $x \geqslant 64$ and Q satisfy $x^{1/3} \ll Q \leqslant (2x)^{1/2}$. Then*

$$\begin{aligned} G_{a,r}(x, Q) &= O(x^{1/2}Q^{a-1/2} + x^{r-1}Q^{a+4-3r}) && \text{if } a > 3r-4, \\ &= O(x^{1/2}Q^{3r-9/2} + x^{r-1}\log x) && \text{if } a = 3r-4, \\ &= O(x^{1/2}Q^{a-1/2}) && \text{if } \tfrac{1}{2} < a < 3r-4, \\ &= O(x^{1/2}\log x) && \text{if } a = \tfrac{1}{2}, \\ &= O(x^{1/2}) && \text{if } O \leqslant a < \tfrac{1}{2}. \end{aligned}$$

THEOREM 2. *Let $r \geqslant 2$ and Q satisfy $x^{2/5} \ll Q \leqslant x^{1/2}$. Then*

$$\begin{aligned} G_{a,r}(x, Q) &= O(Q^{a-1/2}x^{1/2}) && \text{if } a > \tfrac{1}{2}, \\ &= O(x^{1/2}\log Q) && \text{if } a = \tfrac{1}{2} \\ &= O(x^{(4a+3)/10}) && \text{if } O \leqslant a < \tfrac{1}{2}. \end{aligned}$$

It may be noted that Theorem 1 shows the validity of the conjecture (2) in case $a \geqslant \frac{1}{2}$, $r \geqslant 2$ with $\varepsilon = 0$ even (with x^{ε} replaced by $\log x$ in case $a = \frac{1}{2}$) while Theorem 2 gives a non-trivial result for $0 \leqslant a < \frac{1}{2}$.

Proof of Theorem 1. As usual we write $e(x) = \exp(2\pi i x)$ and recall the well-known Fourier expansion (cf. [5, p. 15]):

$$B_r(\{x\}) = -\frac{r!}{(2\pi i)^r} \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \frac{e(mx)}{m^r}.$$

We shall also require the following result due to I. M. Vinogradov [8] concerning exponential sums.

LEMMA. *Let $x \geqslant 64$, A be a real and the reals P, Q satisfy $2x^{1/3} \leqslant P \leqslant Q \leqslant (2x)^{1/2}$. Then there exists a constant N independent of x, A, P and Q such that*

$$\left| \sum_{P<n\leqslant Q} e\left(\frac{x}{n} - An\right) \right| < N\left(\frac{x}{P}\right)^{1/2}.$$

Now let $2^6 < X = X(x) \ll x^{1/2}$, where the function X will be chosen suitably later. Then for $r \geqslant 2$ and any number Q satisfying $(xX)^{1/3} \ll Q \leqslant (2x)^{1/2}$, we have

$$G_{a,r}(x, Q) = -r!(2\pi i)^{-r} \sum_{0<|m|\leqslant 2^{-6}X} m^{-r} \sum_{1\leqslant n\leqslant Q} n^a e\left(\frac{mx}{n}\right) + O\left(X^{1-r} \sum_{1\leqslant n\leqslant Q} n^a\right).$$

For each m satisfying $0 < |m| \leqslant 2^{-6}X$, we choose an integer Y_m subject to the condition $2(2^{-6}x\,|m|)^{1/3} \leqslant Y_m \ll (x\,|m|)^{1/3}$. Also we write

$$\begin{aligned} G_{a,r}(x, Q) &= -\frac{r!}{(2\pi i)^r} \sum_{0<|m|<2^{-6}X} m^{-r} \left(\sum_{1\leqslant n\leqslant Y_m - 1} + \sum_{Y_m \leqslant n \leqslant Q} \right) \\ &\quad \times n^a e\left(\frac{mx}{n}\right) + O\left(\frac{Q^{a+1}}{X^{r-1}}\right) \\ &= -r!(2\pi i)^{-r}(S_1 + S_2) + O(Q^{a+1}X^{1-r}). \end{aligned} \tag{3}$$

It is clear that $(x\,|m|)^{1/3} \ll (xX)^{1/3} \ll Q \leqslant (2x)^{1/2} \leqslant (2x\,|m|)^{1/2}$ and consequently

$$\begin{aligned} S_1 &= O\left(\sum_{0<|m|\leqslant 2^{-6}X} |m|^{-r}\, Y_m^{a+1}\right) = O\left(x^{(a+1)/3} \sum_{1\leqslant m\leqslant X} m^{-r+(a+1)/3}\right) \\ &= O(x^{(a+1)/3}) \qquad \text{if } a < 3r-4, \\ &= O(x^{(a+1)/3}\log x) \qquad \text{if } a = 3r-4, \\ &= O(x^{(a+1)/3}X^{(a+4-3r)/3}) \qquad \text{if } a > 3r-4. \end{aligned} \tag{4}$$

For the estimation of S_2, we write $S_m(n) = \sum_{n\leqslant k\leqslant Q} e(mx/k)$ for each n lying in the interval $[Y_m, Q]$. Then, by Lemma 1, we find $S_m(n) = O(\sqrt{x|m|n^{-1}})$. Consequently, the theorem of partial summation yields for $a \geqslant 0$

$$\begin{aligned}
&\sum_{Y_m \leqslant n \leqslant Q} n^a e\left(\frac{mx}{n}\right) \\
&\quad = Y_m^a S_m(Y_m) + O\left(\sum_{Y_m+1 \leqslant n \leqslant Q} n^{a-1} |S_m(n)|\right) \\
&\quad \ll Y_m^a (x|m| Y_m^{-1})^{1/2} + \sum_{Y_m \leqslant n \leqslant Q} n^{a-1} (x|m| n^{-1})^{1/2} \\
&\quad \ll (x|m|)^{1/2} \left(Y_m^{(2a-1)/2} + \sum_{n \leqslant Q} n^{a-3/2}\right) \\
&\quad \ll (x|m|)^{(a+1)/3} + \begin{cases} (x|m|)^{1/2} & \text{if } 0 \leqslant a < \frac{1}{2}, \\ (x|m|)^{1/2} \log Q & \text{if } a = \frac{1}{2}, \\ (x|m|)^{1/2} Q^{a-1/2} & \text{if } a > \frac{1}{2}. \end{cases}
\end{aligned}$$

Thus from (3), we conclude that for $a > \frac{1}{2}$

$$\begin{aligned}
S_2 &= O\left(\sum_{1 \leqslant m \leqslant X} m^{-r} (x|m|)^{(a+1)/3}\right) + O\left(x^{1/2} Q^{a-1/2} \sum_{1 \leqslant m \leqslant X} m^{1/2-r}\right) \\
&= O(x^{1/2} Q^{a-1/2}) && \text{if } a < 3r-4, \\
&= O(x^{(a+1)/3} \log X + x^{1/2} Q^{a-1/2}) && \text{if } a = 3r-4, \\
&= O(x^{(a+1)/3} X^{(a+4-3r)/3} + x^{1/2} Q^{a-1/2}) && \text{if } a > 3r-4,
\end{aligned} \tag{5}$$

and that

$$\begin{aligned}
S_2 &= O(x^{(a+1)/3} + x^{1/2} \log Q) = O(x^{1/2} \log x) && \text{if } a = \tfrac{1}{2}, \\
&= O(x^{1/2}) && \text{if } 0 \leqslant a < \tfrac{1}{2}.
\end{aligned} \tag{6}$$

Hence, on collecting (3), (4), (5) and (6), we have

$$\begin{aligned}
&G_{a,r}(x, Q) \\
&\quad = O(x^{1/2} + Q^{a+1} X^{1-r}) && \text{if } 0 \leqslant a < \tfrac{1}{2}, \\
&\quad = O(x^{1/2} \log x + Q^{a+1} X^{1-r}) && \text{if } a = \tfrac{1}{2}, \\
&\quad = O(x^{1/2} Q^{a-1/2} + Q^{a+1} X^{1-r}) && \text{if } \tfrac{1}{2} < a < 3r-4, \\
&\quad = O(x^{(a+1)/3} \log X + x^{1/2} Q^{a-1/2} \\
&\qquad\qquad + Q^{a+1} X^{1-r}) && \text{if } a = 3r-4, \\
&\quad = O(x^{(a+1)/3} X^{(a+4-3r)/3} + x^{1/2} Q^{a-1/2} \\
&\qquad\qquad + Q^{a+1} X^{1-r}) && \text{if } a > 3r-4.
\end{aligned}$$

Now, choosing X to be a constant multiple of $(Q^3/x)^{1/2(r-1)}$ in case $0 \leqslant a \leqslant 3r-4$ and of $\frac{Q^3}{x}$ in case $a > 3r-4$, we see that the conclusions of Theorem 1 hold.

Proof of Theorem 2. The proof is based on the following two estimates which follow, respectively, from Theorems 5.9 and 5.11 of Titchmarsh [7]: Let $z > 0$ and the reals $M > 0$ and M' be subject to $M + 1 \leqslant M' \leqslant 2M$. Then

$$\sum_{M<n\leqslant M'} e\left(\frac{z}{n}\right) = O(M^{-1/2}z^{1/2} + M^{3/2}z^{-1/2}), \tag{7}$$

$$\sum_{M<n\leqslant M'} e\left(\frac{z}{n}\right) = O(M^{1/3}z^{1/6} + M^{7/6}z^{-1/6}), \tag{8}$$

where the order constants are absolute.

Now let $1 \leqslant P = P(x) \leqslant Q$ be a function of x to be chosen suitably later. We write

$$G_{a,r}(x, Q) = -\frac{r!}{(2\pi i)^r} \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} (S_1(m) + S_2(m))\, m^{-r}, \tag{9}$$

where

$$S_1(m) = \sum_{1\leqslant n\leqslant P} n^a e\left(\frac{mx}{n}\right), \qquad S_2(m) = \sum_{P<n\leqslant Q} n^a e\left(\frac{mx}{n}\right).$$

To estimate $S_2(m)$, we introduce $S(M) = \sum_{M\leqslant n\leqslant Q} e(mx/n)$ for $M \leqslant Q$. We split the interval $[M, Q]$ into subintervals $[M, 2M]$, $(2M, 2^2M]$,..., $(2^{s-1}M, 2^sM]$ and $(2^sM, Q]$, where $s = -[-\log(Q/M)/\log 2] - 1$. Then using (7) to estimate the sum over each of these subintervals, we conclude that

$$\begin{aligned} S(M) &= O\left(M^{-1/2}|mx|^{1/2} \sum_{i=0}^{s} (2^{-1/2})^i + |mx|^{-1/2} (2^sM)^{3/2} \sum_{i=0}^{s} (2^{-3/2})^{s-i}\right) \\ &= O(M^{-1/2}|mx|^{1/2} + |mx|^{-1/2} Q^{3/2}). \end{aligned}$$

Now by the theorem of partial integration for Stieltjes integrals and by standard theorems for converting sums to Stieltjes integrals, we have for $a \neq 0$

$$\begin{aligned} S_2(m) &= a\int_P^Q t^{a-1}S(t)\,dt - a\int_{[Q]}^Q t^{a-1}S(t)\,dt + S([P]+1)\,P^a \\ &= O\left(|mx|^{1/2}\int_P^Q t^{a-3/2}\,dt + |mx|^{-1/2}\,Q^{3/2}\int_P^Q t^{a-1}\,dt\right) \\ &\quad + O(Q^{a-1}) + O(|mx|^{1/2}\,P^{a-1/2} + |mx|^{-1/2}\,Q^{3/2}P^a) \\ &= O\left(|mx|^{1/2}(Q^{a-1/2} + P^{a-1/2})\left(\log\frac{Q}{P}\right)^{\delta_{a,1/2}}\right. \\ &\quad \left. + |mx|^{-1/2}\,Q^{3/2+a}\right). \end{aligned}$$

Since $S_2(m) = S([P]+1)$ for $a = 0$, we have altogether

$$\begin{aligned} S_2(m) &= O(Q^{a+3/2}|mx|^{-1/2}) \\ &\quad + O(Q^{a-1/2}(\log x)^{\delta_{a,0}}|mx|^{1/2}) \qquad \text{if } a \geqslant \tfrac{1}{2}, \\ &\quad + O(P^{a-1/2}|mx|^{1/2}) \qquad \text{if } 0 \leqslant a < \tfrac{1}{2}. \end{aligned} \tag{10}$$

To estimate $S_1(m)$ we introduce $T(M) = \sum_{M \leqslant n \leqslant P} e(mx/n)$ for $M \leqslant P$. This time, using (8) and arguing as before, we conclude that

$$\begin{aligned} T(M) &= O\left(|mx|^{1/6} P^{1/3} \sum_{i \ll \log P} (2^{-1/3})^i + |mx|^{-1/6} P^{7/6} \sum_{i \ll \log P} (2^{-7/6})^i\right) \\ &= O(|mx|^{1/6} P^{1/3} + |mx|^{-1/6} P^{7/6}). \end{aligned}$$

Hence by using the theorem of partial summation, we have for $a \neq 0$

$$\begin{aligned} S_1(m) &= a \int_1^P T(t)\, t^{a-1}\, dt + T(1) + O(P^{a-1}) \\ &= O\left[\left(P^{1/3}|mx|^{1/6} + P^{7/6}|mx|^{-1/6}\right)\int_1^P t^{a-1}\, dt\right)\bigg] \\ &\quad + O(P^{1/3}|mx|^{1/6} + P^{7/6}|mx|^{-1/6}). \end{aligned}$$

Since the case $a = 0$ is included in the above estimation (because $S_1(m) = T(1)$ in that case), we have

$$S_1(m) = O(P^{a+1/3}|mx|^{1/6} + P^{a+7/6}|mx|^{-1/6}). \tag{11}$$

Thus from (9), (10) and (11) it follows that

$$\begin{aligned} G_{a,r}(x, Q) &= O\left(P^{a+1/3}x^{1/6} \sum_{m=1}^{\infty} m^{1/6-r} + P^{a+7/6}x^{-1/6} \sum_{m=1}^{\infty} m^{-1/6-r} \right. \\ &\quad \left. + x^{-1/2}Q^{a+3/2} \sum_{m=1}^{\infty} m^{-1/2-r}\right) \\ &\quad + \begin{cases} O\left(x^{1/2}Q^{a-1/2}(\log x)^{\delta_{a,1/2}} \sum_{m=1}^{\infty} m^{1/2-r}\right) & \text{if } a \geqslant \tfrac{1}{2}, \\ O\left(x^{1/2}P^{a-1/2} \sum_{m=1}^{\infty} m^{1/2-r}\right) & \text{if } 0 \leqslant a < \tfrac{1}{2}, \end{cases} \\ &= O(P^{a+1/3}x^{1/6} + P^{a+7/6}x^{-1/6} + x^{-1/2}Q^{a+3/2}) \\ &\quad + \begin{cases} O(x^{1/2}Q^{a-1/2}(\log x)^{\delta_{a,1/2}}) & \text{if } a \geqslant \tfrac{1}{2}, \\ O(x^{1/2}P^{a-1/2}) & \text{if } 0 \leqslant a < \tfrac{1}{2}. \end{cases} \end{aligned}$$

Now, choosing $P = P(x)$ to be a constant multiple of $x^{2/5}$, we see that the conclusions of Theorem 2 hold.

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